A nonMarkov model of cavity radiation with thermal characteristics

1986 J. Phys. A: Math. Gen. 19 L513

(http://iopscience.iop.org/0305-4470/19/9/009)

J. Phys. A: Math. Gen. 19 (1986) L513-L516. Printed in Great Britain

LETTER TO THE EDITOR

A non-Markov model of cavity radiation with thermal characteristics

S K Srinivasan
Department of Mathematics, Indian Institute of Technology, Madras 600 036, India

Received 10 March 1986

Abstract. A model of cavity radiation is proposed in which the stimulated emission is modelled as an age-dependent birth process. The photon population statistics, under certain special conditions, are shown to possess thermal characteristics to second order.

The problem of cavity field evolution and its approach to thermal equilibrium was considered long ago by Shimoda *et al* [1] who used a population growth model as the basis. However the concept of population inversion and the various consequences that would follow from it were somehow derived quite independently (see [2, 3]) and it is only recently that interest has been revived in population point processes [4-9]. The model in fact provided a great stimulus for the development of the rate equations and a semiclassical approach to the theory of amplifiers in general [4, 5]. More recently Shepherd [10] put forward an integrated approach by taking into account the continual nature of the cavity-field and field-detector interactions. More specifically he established that the Markov nature of the evolution implied the Gaussian-Lorentzian nature of the resulting spectrum. The object of this letter is to propose an age-dependent growth model of cavity radiation and show that the photon population statistics can, in spite of non-Markov evolution, exhibit thermal characteristics at least to second order.

We shall use Shepherd's model [10, 11] of cavity radiation and detection as the starting point. The photon field is modelled as a discrete-valued stochastic population process; the evolution of the field in the cavity is modelled as a birth, death and immigration process. The field-detector interaction is modelled as an emigration process with a constant rate η per individual. In the model used by Shepherd, the birth, death and immigration rates are constants equal respectively to λ, μ and ν. The parameter λ is identified to be the rate of stimulated emission and ν is that of spontaneous emission. If $\lambda = \nu$, the model exactly describes Gaussian light with a Lorentzian spectral profile. To this contribution we make a modification by letting the population evolve in a non-Markov manner. The motivation to use dependence on memory effects stems from a general hope that inhibitive effects which can lead to reduction in bunching can be accomodated within the framework of a non-Markov (memory-dependent) model. More specifically, we assume that the birth rate is not a constant; we of course keep death and immigration rates as constants equal to μ and ν respectively. The field-detector interaction is also modelled in the same way as in the Shepherd model [10]. We model the birth process as an age-dependent process in which the birth rate decreases with age.

Although the analysis of age-dependent population growth is rather intractable in its most general form [12, 13], the moments of the population size can be estimated [14] for some specific age-dependent rates. Even so, the analysis is rather cumbersome and hence we propose an entirely new way of incorporating the age dependence. Each of the cavity photons, conditional upon its survival of death (absorption) and detection, is assumed to have a lifespan consisting of three phases. In the first phase the photon passively interacts with the cavity resulting in no emissions at all. In the second phase the photon is very active and stimulated emissions take place at a rate a while in the final phase it is again inactive with no emissions taking place. The last phase corresponds to the situation when the photon is out of the main interaction region and hence no amplification is possible. We assume that the lifespans of the phases are independent and that the spans of the first two phases are exponentially distributed with parameter λ. The span of the third phase is indefinite; however it should be noted that the photon will eventually be absorbed or removed by detection. It follows that if the total lifespan x of a photon is conditional upon its survival of absorption and detection, the rate of birth (stimulated emission) is $e^{-\lambda x}\lambda a x$. In fact a nice connection can be established between the model based on age-dependent birth rate and the one we have proposed but we shall not go into the details here.

Next we observe that there are two alternatives available according to whether photons in phase 3 can be detected or not. In this letter we shall assume that the photons in phase 3 cannot be detected and hence go out of our view. We further assume that the immigration process creates photons in phase 1. We denote by $X(t)$ and $Y(t)$ respectively the number of photons at time t in phases 1 and 2 and introduce the probability generating functions $g(z_1, z_2, t)$ and $g_i(z_1, z_2, t)$ $(i = 1, 2)$ by

$$g_i(z_1, z_2, t) = E[z_1^{X(t)} z_2^{Y(t)} | X(0) = 2 - i,\ Y(0) = i - 1,\ \nu = 0] \tag{1}$$

$$g(z_1, z_2, t) = E[z_1^{X(t)} z_2^{Y(t)} | X(0) = Y(0) = 0,\ \nu \neq 0] \tag{2}$$

where E stands for the expectation of the quantity in square brackets. The analysis is essentially on very similar lines to those in [15] and [16]. Full details will be presented elsewhere and we give below the main equations and the results pertaining to the statistics of photon detection.

The exponential nature of the lifespan distribution of the first two phases and the branching nature of the process as a whole lead to the following differential equations:

$$\frac{\partial g_1(z_1, z_2, t)}{\partial t} = -(\lambda + \eta + \mu) g_1(z_1, z_2, t) + \lambda g_2(z_1, z_2, t) + \mu + \eta \tag{3}$$

$$\frac{\partial g_2(z_1, z_2, t)}{\partial t} = -(\lambda + \eta + \mu + a) g_2(z_1, z_2, t) + a g_2(z_1, z_2, t) g_1(z_1, z_2, t) + \lambda + \mu + \eta \tag{4}$$

$$\frac{\partial g(z_1, z_2, t)}{\partial t} = -\nu g(z_1, z_2, t) + \nu g(z_1, z_2, t) g_1(z_1, z_2, t) \tag{5}$$

with the initial conditions

$$g_i(z_1, z_2, 0) = z_i \qquad g(z_1, z_2, 0) = 1. \tag{6}$$

We next introduce the moments $A_j^i(t)$, $B_k^{ij}(t)$, $A^i(t)$, $B^{ij}(t)$ $(i, j, k = 1, 2)$ by

$$A_j^i(t) = \left.\frac{\partial g_j}{\partial z_i}\right|_{z_1 = z_2 = 1} \qquad B_k^{ij}(t) = \left.\frac{\partial^2 g_k}{\partial z_i \partial z_j}\right|_{z_1 = z_2 = 1} \tag{7}$$

$$A^i(t)=\frac{\partial g}{\partial z_i}\bigg|_{z_1=z_2=1} \qquad B^{ij}(t)=\frac{\partial^2 g}{\partial z_i \partial z_j}\bigg|_{z_1=z_2=1}. \tag{8}$$

The moments can be calculated by differentiating the equations (3)-(5) and solving the resulting linear differential equations.

The photon detection process is described by the sequence of product densities introduced in [15]. If $N(t)$ is the number of photons detected over the interval $[0, t]$, then the product densities are given by

$$h_1^i(t)=\lim_{\Delta\to 0} \mathrm{pr}\{N(t+\Delta)-N(t)=1|X(0)=2-i,\ Y(0)=i-1,\ \nu=0\}/\Delta \tag{9}$$

$$h_1(t)=\lim_{\Delta\to 0} \mathrm{pr}\{N(t+\Delta)-N(t)=1|X(0)=Y(0)=0,\ \nu\neq 0\}/\Delta. \tag{10}$$

Second-order product densities are defined in a similar way. If we denote by $h^{\mathrm{sty}}(t)$ the stationary value of the second-order product density of detection, then we have

$$h^{\mathrm{sty}}(t)=h_1(\infty)h_1(t)+\sum_{i,j} B^{ij}(\infty)h_1^j(t)\eta. \tag{11}$$

The function $h_1^j(t)$ can be directly computed once we know the moments:

$$h_1^j(t)=\eta[A_j^1(t)+A_j^2(t)] \tag{12}$$

$$h_1(t)=\nu\int_0^t h_1^1(t')\,\mathrm{d}t'. \tag{13}$$

The moments can be calculated and the relevant quantities of interest are given by

$$B^{11}(\infty)=[\nu^2(\lambda+\mu+\eta)^2+\tfrac{1}{2}\nu\lambda a^2]/D$$

$$B^{22}(\infty)=\lambda^2\nu(\nu+\tfrac{1}{2}a)/D \qquad B^{21}(\infty)=B^{12}(\infty)=\nu\lambda(\lambda+\mu+\eta)(\nu+\tfrac{1}{2}a)/D \tag{14}$$

$$h_1^1(t)=\tfrac{1}{2}\eta\{[(\lambda/a)^{1/2}+1]p(t)-[(\lambda/a)^{1/2}-1]q(t)\} \tag{15}$$

$$h_1^2(t)=\tfrac{1}{2}\eta\{[(a/\lambda)^{1/2}+1]p(t)-[(a/\lambda)^{1/2}-1]q(t)\} \tag{16}$$

where

$$p(t)=\exp\{-[\lambda+\mu+\eta-(\lambda a)^{1/2}]t\} \qquad q(t)=\exp\{-[\lambda+\mu+\eta+(\lambda a)^{1/2}]t\} \tag{17}$$

$$D=[(\lambda+\mu+\eta)^2-\lambda a]^2. \tag{18}$$

We set $\lambda=a$ and substitute in (11) the values for $B^{ij}(\infty)$ from (14) to obtain

$$h^{\mathrm{sty}}(t)=\frac{\nu\eta^2}{(\mu+\eta)^2}\left(\nu+\frac{\lambda^2}{2\lambda+\mu+\eta}\exp[-(\mu+\eta)t]\right). \tag{19}$$

If at this stage we introduce the measure $\mathscr{B}$ of bunching by

$$\mathscr{B}=h^{\mathrm{sty}}(0)/h^{\mathrm{sty}}(\infty) \tag{20}$$

we find

$$\mathscr{B}=1+\lambda^2/[\nu(2\lambda+\mu+\eta)] \tag{21}$$

from which we conclude that $\mathscr{B}$ can be made close to 1 by choosing λ small. If on the other hand we choose $\nu=\lambda^2/(2\lambda+\mu+\eta)$, we find $\mathscr{B}=2$ and (19) takes the form

$$h^{\mathrm{sty}}(t)=\frac{\nu^2\eta^2}{(\mu+\eta)^2}\{1+\exp[-(\mu+\eta)t]\}. \tag{22}$$

We note that within the framework of Gaussian light, the autocorrelation of the incident light beam is related to the detection process by [17]

$$h^{\text{sty}}(t) = [\bar{I}^2 + |R(t)|^2]\eta^2 \tag{23}$$

where $\bar{I}$ is the mean intensity and $R(t)$ is the stationary second-order correlation of the incident beam. Hence we have the identification

$$R(t) = [\nu/(\mu+\eta)] \exp[-\tfrac{1}{2}(\mu+\eta)t] \tag{24}$$

enabling us to conclude that the choice of ν and λ results in a thermal light beam with a Lorentzian profile. Normally we expect a Markov model to lead to a Lorentzian profile as did happen for the Shepherd model. In our model λ represents the rate of loss of photons to the third phase and $a = \lambda$ restores some kind of balance. However the choice $\nu = \lambda^2/(2\lambda + \mu + \eta)$ is rather difficult to interpret directly. At the moment the only way is to regard this choice as some kind of a renormalisation of the rate of stimulated emission. Full details and features relating to the case when photons in phase 3 are also subject to detection will be published elsewhere [18].

References

[1] Shimoda K, Takahasi H and Townes C H 1957 *J. Phys. Soc. Japan* **12** 686–700
[2] Townes C H 1984 *IEEE J. Quantum Electron.* **QE-20** 547–50
[3] Lamb W E 1984 *IEEE J. Quantum Electron.* **QE-20** 551–5
[4] Lax M 1967 *IEEE J. Quantum Electron.* **QE-3** 37–46
[5] Haken H 1970 *Laser Theory* (Berlin: Springer)
[6] Schell A and Barakat R 1973 *J. Phys. A: Math. Gen.* **6** 826–36
[7] Huang X Y, Narducci L M and Yuan J M 1981 *Phys. Rev.* A **23** 3084–93
[8] Saleh B E A, Stoler D and Teich M C 1983 *Phys. Rev.* A **27** 360–74
[9] Teich M C, Saleh B E A and Perina J 1984 *J. Opt. Soc. Am.* B **1** 366–86
[10] Shepherd T J 1981 *Opt. Acta* **28** 567–83
[11] Jakeman E and Shepherd T J 1984 *J. Phys. A: Math. Gen.* **17** 745–7
[12] Bartlett M S 1966 *Stochastic Processes* 2nd edn (Cambridge: Cambridge University Press)
[13] Harris T E 1963 *Theory of Branching Processes* (Berlin: Springer)
[14] Srinivasan S K and Rao N V K 1968 *J. Math. Anal. Appl.* **21** 43–52
[15] Srinivasan S K and Vasudevan R 1985 *Opt. Acta* **32** 749–66
[16] Srinivasan S K and Vasudevan R 1986 *Opt. Acta* **33** 191–205
[17] Srinivasan S K and Vasudevan R 1967 *Nuovo Cimento* **47** 185–93
[18] Srinivasan S K 1986 *Pramana* submitted